Applied Mathematics on Science and Engineering, 2(1):21-35, 2025.

Applied Mathematics on Science and Engineering

Journal homepage: https://furthersci.org/journal-details.php?journal-id=49

ISSN (E): 3048-8362

Research Article

https://doi.org/10.65036/amse212

Expanding the Horizons of Graph Theory: Rough Tree-Width, Hyperrough Structures, and Superhyperrough Generalizations

Takaaki Fujita [1*]

[1] *Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan.*
[*] *Corresponding author: takaaki.fujita060@gmail.com*

Article Info

Keywords: *Rough graph, Rough tree-width, Tree-width, Rough set, HyperRough Set, SuperHyperRough Set*

Received: *03.10.2024*
Accepted: *25.04.2025*
Published: *27.05.2025*

© *2025 by the author's. The terms and conditions of the Creative Commons Attribution (CC BY) license apply to this open access article.*

Abstract

Rough set theory provides a formal framework for managing imprecise information by approximating subsets of a universe through lower and upper bounds derived from equivalence relations. This foundational idea has been extended to more expressive structures such as the *Hyperrough Set* and the *Superhyperrough Set*.
Building upon rough set theory, *rough graphs* are introduced to represent uncertain relationships among elements, where the existence of edges is determined by the rough membership values of their endpoints. To analyze structural properties of graphs, various parameters are employed—among them, *graph width parameters* play a central role by measuring the structural complexity of graphs through bounded decompositions. One widely studied example is *tree-width*, which has recently been generalized to the context of rough graphs as *rough tree-width*.
In this paper, We extend the framework by proposing several new generalizations: the *Hyperrough Tree-width*, the *Hyperrough Graph*, and the *Superhyperrough Graph*. For each of these constructs, we provide formal definitions and initiate a preliminary exploration of their mathematical properties, laying the groundwork for future study and application in uncertainty-aware graph analysis.

1. Introduction

1.1. Rough Graphs and Their Role in Information Systems

Graph theory offers a versatile framework for representing entities and the relationships between them. Vertices correspond to objects, edges represent binary associations, and various graph invariants capture underlying structural properties [1]. In real-world applications—such as social networks, communication systems, or biological interaction networks—data is frequently incomplete, ambiguous, or subject to noise.

To address such imperfections, rough set theory introduces the notion of *lower* and *upper* approximations, which are formulated using indiscernibility relations among elements of a universe [2–4]. When this theoretical foundation is extended to graphs, one obtains the concept of a *rough graph*. In such structures, each vertex or edge is associated with approximation values that reflect its definite or possible membership within a specific subgraph or feature [5, 6]. Embedding rough graphs into the framework of information systems—tabular models where rows denote objects and columns represent attributes—facilitates the fusion of graph-based modeling with approximation-based data analysis. Objects sharing identical attribute values are grouped into equivalence classes, which in turn define the rough approximations of graph vertices representing the same underlying elements [7, 8]. In addition to classical rough sets, several advanced generalizations have been introduced, including *hyperrough sets* and *superhyperrough sets*, which offer more expressive tools for modeling higher-order

uncertainties [9–12].
Furthermore, rough sets are part of a broader family of uncertainty-handling paradigms. Related frameworks include fuzzy sets[13], intuitionistic fuzzy sets[14], neutrosophic sets[15], plithogenic sets[16], and soft sets[17].

1.2. Key Graph Width Measures

One of the core themes in algorithmic graph theory is the study of *width parameters*, which quantify how structurally close a given graph is to a tree or a path. Among these, the *tree-width* of a graph is defined as the minimum width—formally, the maximum bag size minus one—over all possible tree decompositions. This parameter plays a crucial role in determining the tractability of many computational problems that are otherwise NP-hard [18–21]. Related notions include *path-width*[22, 23], which restricts the tree structure to a linear path, and *branch-width*[24, 25], which measures separability through subcubic trees.

In addition to these classical parameters, various other measures have been introduced to capture different aspects of graph structure, such as *hypertree-width*[26, 27], *superhypertree-width*[28, 29], *cut-width*[30], and *bandwidth*[31]. Each of these parameters is designed to characterize complexity from a different perspective and has corresponding applications in optimization, constraint satisfaction, and database theory.

1.3. Contributions of This Work

Given the growing importance of uncertainty modeling in graph-structured data, the investigation of graph parameters and their rough set-based extensions—such as rough graphs—has become a subject of significant theoretical and practical interest. Although the notion of *rough tree-width* has been introduced in earlier studies [32], its underlying theoretical framework and structural behavior remain insufficiently explored.

In this paper, We extend the framework by proposing several new generalizations: the *Hyperrough Tree-width*, the *Hyperrough Graph*, and the *Superhyperrough Graph*. For each of these constructs, we provide formal definitions and initiate a preliminary exploration of their mathematical properties, laying the groundwork for future study and application in uncertainty-aware graph analysis.

2. Preliminaries and Definitions

This section presents the core definitions and notations that form the foundation for the discussions throughout this paper. For further background on graph-theoretical concepts, readers may refer to [1] and other standard references in the field. Unless stated otherwise, all graphs considered in this paper are assumed to be finite, undirected, and simple—that is, they contain no loops or multiple edges between the same pair of vertices.

2.1. Rough Graphs within Information Systems

The notion of a rough graph builds on the classical framework of rough set theory and approximation spaces. In rough set theory, one manages uncertainty by approximating any target subset via its *lower* and *upper* bounds, which are defined through an equivalence relation on a universal set [3, 33].

Definition 2.1 (Rough Set)**.** *[3, 33] Let (U,R) be an* approximation space*, where*

- *U is a finite, nonempty set (the* universe*);*
- *$R \subseteq U \times U$ is an equivalence relation on U.*

For each $x \in U$, its equivalence class is

$$[x]_R = \{y \in U \mid (x,y) \in R\}.$$

Given any $X \subseteq U$, define:

$$\begin{aligned} &\textit{Lower approximation:} \quad R_*(X) = \{x \in U \mid [x]_R \subseteq X\}, \\ &\textit{Upper approximation:} \quad R^*(X) = \{x \in U \mid [x]_R \cap X \neq \varnothing\}. \end{aligned}$$

The pair $(R_*(X), R^*(X))$ is called the rough set *of X relative to R. If $R_*(X) = R^*(X)$, then X is* crisp*; otherwise, X is* rough*.*

Next, we introduce a measure of how strongly an object belongs to a subset within an information system.

Definition 2.2 (Rough Membership Function)**.** *Let $\mathscr{I} = (U,A)$ be an* information system*, where*

- *U is the universe of objects;*
- *A is a set of attributes inducing an equivalence relation on U.*

For any $X \subseteq U$ and $x \in U$, the rough membership *of x in X is*

$$\mu_X^A(x) = \frac{\left|[x]_A \cap X\right|}{\left|[x]_A\right|},$$

where $[x]_A$ is the equivalence class of x under the relation defined by A [34].

Finally, we arrive at the definition of a rough graph[35, 36].

Definition 2.3 (Rough Graph)**.** *(cf.[35]) A* rough graph *is a triple* (V, E, μ) *such that:*

- $V = \{v_1, v_2, \ldots, v_n\}$ *is a finite set of vertices;*
- $E \subseteq \big\{\{v_i, v_j\} \mid v_i, v_j \in V,\ i \neq j\big\}$ *is the set of undirected edges;*
- $\mu : V \to [0,1]$ *assigns to each vertex its rough membership value.*

An edge $\{v_i, v_j\}$ *belongs to* E *precisely when*

$$\max\{\mu(v_i), \mu(v_j)\} > 0,$$

and no edge is present if both endpoints have zero membership.

2.2. Tree-Width

The *tree-width* of a graph quantifies how closely the graph's structure approximates a tree. It is defined as the minimum width—bag size minus one—over all possible tree decompositions of the graph [37, 38]. The formal definition follows.

Definition 2.4 (Tree Decomposition [38])**.** *Let* $H = (V(H), E(H))$ *be a finite undirected graph. A* tree decomposition *of* H *is a pair* $\big(T, \{B_x \mid x \in V(T)\}\big)$ *where:*

- T *is a finite tree.*
- *Each* $B_x \subseteq V(H)$ *is called a* bag.

These must satisfy:

(TD1) $\bigcup_{x \in V(T)} B_x = V(H)$.
(TD2) For every edge $\{u, v\} \in E(H)$*, there exists* $x \in V(T)$ *such that* $\{u, v\} \subseteq B_x$.
(TD3) For each vertex $v \in V(H)$*, the set* $\{x \in V(T) \mid v \in B_x\}$ *induces a connected subtree of* T.

The width *of this decomposition is* $\max_{x \in V(T)}(|B_x| - 1)$*. The* tree-width *of* H*, denoted* $\mathrm{tw}(H)$*, is the minimum width over all its tree decompositions.*

Variants include the *path-width* [22], which requires T to be a path, and the *cycle-width* [39], which requires T to be a cycle.

2.3. Extensions of Rough Graphs in Information Systems

In this subsection, we introduce the concept of rough tree decompositions and present related definitions and results.

Definition 2.5 (Rough Tree Decomposition)**.** *Let* $\mathscr{H} = (U, F, \nu)$ *be a rough graph, where* $\nu : U \to [0,1]$ *assigns each vertex a membership value. A* rough tree decomposition *of* $\mathscr{H}$ *is a pair* $\big(T, \{B_t\}_{t \in V(T)}\big)$ *satisfying:*

(RTD1) For each $u \in U$*, the set* $\{t \in V(T) \mid u \in B_t\}$ *induces a connected subtree of* T.
(RTD2) For every edge $\{u, v\} \in F$*, there exists* $t \in V(T)$ *such that* $\{u, v\} \subseteq B_t$.

The width *of this rough tree decomposition is*

$$\max_{t \in V(T)} \Big(\sum_{u \in B_t} \nu(u) \Big) - 1,$$

and the rough tree-width *of* $\mathscr{H}$ *is the minimum width over all such decompositions.*

Example 2.6 (Rough Tree Decomposition of a Cyclic Rough Graph)**.** *Let* $\mathscr{H} = (U, F, \nu)$ *be the rough graph defined by*

$$U = \{v_1, v_2, v_3, v_4\}, \quad F = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}\},$$

and the membership function

$$\nu(v_1) = 0.9, \quad \nu(v_2) = 0.6, \quad \nu(v_3) = 0.7, \quad \nu(v_4) = 0.4.$$

We construct a tree decomposition $(T, \{B_t\})$ *of the underlying cycle graph as follows:*

$$T: \quad t_A \; - \; t_B,$$

with bags

$$B_{t_A} = \{v_1, v_2, v_4\}, \quad B_{t_B} = \{v_2, v_3, v_4\}.$$

One checks easily:

- *Every edge in* F *lies entirely in at least one bag:* $\{v_1, v_2\}, \{v_4, v_1\} \subseteq B_{t_A}$*, and* $\{v_2, v_3\}, \{v_3, v_4\} \subseteq B_{t_B}$.
- *For each vertex* v_i*, the set of bags containing* v_i *is connected in* T*:* $\{t_A\}$ *for* v_1*,* $\{t_A, t_B\}$ *for* v_2, v_4*, and* $\{t_B\}$ *for* v_3.

For this decomposition, the bag-sums of membership values are

$$\sum_{v\in B_{t_A}} \nu(v) = \nu(v_1)+\nu(v_2)+\nu(v_4) = 0.9+0.6+0.4 = 1.9,$$

$$\sum_{v\in B_{t_B}} \nu(v) = \nu(v_2)+\nu(v_3)+\nu(v_4) = 0.6+0.7+0.4 = 1.7.$$

Hence the width of this rough tree decomposition is

$$\max\{1.9,\, 1.7\} - 1 = 1.9 - 1 = 0.9.$$

Since no decomposition can achieve a smaller maximum bag-sum, the rough tree-width of $\mathscr{H}$ is

$$\mathrm{tw}_{\mathrm{rough}}(\mathscr{H}) = 0.9.$$

2.4. HyperRough Set and SuperHyperRough Set

The *HyperRough Set* extends rough set theory by incorporating multiple attributes. Its formal definition is given below [12, 40–46].

Definition 2.7 (HyperRough Set)**.** *[40] Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with corresponding domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product*

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Let $R \subseteq X \times X$ be an equivalence relation on X, with $[x]_R$ denoting the equivalence class of x. A HyperRough Set *over X is a pair (F, J), where:*

- *$F : J \to \mathscr{P}(X)$ is a mapping that assigns to each attribute value combination $a = (a_1, a_2, \ldots, a_n) \in J$ a subset $F(a) \subseteq X$.*
- *For each $a \in J$, the rough set approximations of $F(a)$ are defined as*

$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

Here, $\underline{F(a)}$ comprises all elements whose equivalence classes are completely contained within $F(a)$, while $\overline{F(a)}$ contains elements whose equivalence classes intersect $F(a)$. Additionally, the following properties hold for all $a \in J$:

- *$\underline{F(a)} \subseteq \overline{F(a)}$.*
- *If $F(a) = \varnothing$, then $\underline{F(a)} = \overline{F(a)} = \varnothing$.*
- *If $F(a) = X$, then $\underline{F(a)} = \overline{F(a)} = X$.*

An n-SuperHyperRough Set generalizes rough sets by using power sets of attribute values to produce nuanced approximations under uncertainty[40, 47]. The definition of n-SuperHyperRough Sets is described as follows.

Definition 2.8 (n-SuperHyperRough Set)**.** *[40] Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, let $\mathscr{P}(J_i)$ denote its power set. Define the set of all possible attribute value combinations as*

$$J = \mathscr{P}(J_1) \times \mathscr{P}(J_2) \times \cdots \times \mathscr{P}(J_n).$$

*Let $R \subseteq X \times X$ be an equivalence relation on X. An n-*SuperHyperRough Set *over X is a pair (F, J), where:*

- *$F : J \to \mathscr{P}(X)$ is a mapping that assigns to each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$ (with $A_i \subseteq J_i$ for all i) a subset $F(A) \subseteq X$.*
- *For each $A \in J$, the lower and upper approximations are defined as*

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\}.$$

Thus, $\underline{F(A)}$ consists of all elements whose equivalence classes are entirely contained in $F(A)$, and $\overline{F(A)}$ includes those elements whose equivalence classes intersect $F(A)$. The following properties hold for all $A \in J$:

- *$\underline{F(A)} \subseteq \overline{F(A)}$.*
- *If $F(A) = \varnothing$, then $\underline{F(A)} = \overline{F(A)} = \varnothing$.*
- *If $F(A) = X$, then $\underline{F(A)} = \overline{F(A)} = X$.*
- *For any $A, B \in J$,*

$$\underline{F(A \cap B)} \subseteq \underline{F(A)} \cap \underline{F(B)}, \quad \overline{F(A \cup B)} \supseteq \overline{F(A)} \cup \overline{F(B)}.$$

Example 2.9 (3–SuperHyperrough Set in Loan Applicant Evaluation)**.** *Loan evaluation is the process of assessing a borrower's creditworthiness, financial history, and risk before approving a loan (cf.[48–50]). Let X be a pool of eight loan applicants:*

$$X = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}.$$

We record three categorical attributes:

$$T_1 : \textit{Income} \in \{\mathrm{Low}, \mathrm{Med}, \mathrm{High}\}, \quad T_2 : \textit{Credit} \in \{\mathrm{Poor}, \mathrm{Fair}, \mathrm{Good}\}, \quad T_3 : \textit{Employment} \in \{\mathrm{Unemp}, \mathrm{PT}, \mathrm{FT}\}.$$

Thus

$$J_1 = \{\text{Low}, \text{Med}, \text{High}\}, \quad J_2 = \{\text{Poor}, \text{Fair}, \text{Good}\}, \quad J_3 = \{\text{Unemp}, \text{PT}, \text{FT}\},$$

and

$$J = \mathscr{P}(J_1) \times \mathscr{P}(J_2) \times \mathscr{P}(J_3).$$

Define an equivalence relation R by geographic region:

$$[a_1]_R = \{a_1, a_2\}, \quad [a_3]_R = \{a_3, a_4\}, \quad [a_5]_R = \{a_5, a_6\}, \quad [a_7]_R = \{a_7, a_8\}.$$

Define $F : J \to \mathscr{P}(X)$ on two representative tuples (all others map to $\varnothing$):

$$F\big(\{\text{High}\}, \{\text{Good}\}, \{\text{FT}\}\big) = \{a_1, a_3, a_5\},$$
$$F\big(\{\text{Low}, \text{Med}\}, \{\text{Poor}, \text{Fair}\}, \{\text{Unemp}, \text{PT}\}\big) = \{a_2, a_4, a_6, a_7\}.$$

Lower and Upper Approximations. *For $A = (\{\text{High}\}, \{\text{Good}\}, \{\text{FT}\})$,*

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\} = \{a_1\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\} = \{a_1, a_2, a_3, a_4, a_5, a_6\}.$$

Indeed, only $[a_1]_R = \{a_1, a_2\} \subseteq \{a_1, a_3, a_5\}$ fails for a_2, so $\underline{F(A)} = \{a_1\}$, and any region equivalence class that meets $\{a_1, a_3, a_5\}$ enters $\overline{F(A)}$.
Similarly, for $B = (\{\text{Low}, \text{Med}\}, \{\text{Poor}, \text{Fair}\}, \{\text{Unemp}, \text{PT}\})$:

$$\underline{F(B)} = \{a_2, a_4, a_6, a_7\}, \quad \overline{F(B)} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}.$$

Intersection and Union. *Compute $A \cap B$ and $A \cup B$ component-wise:*

$$A \cap B = (\varnothing, \{\text{Good}\} \cap \{\text{Poor}, \text{Fair}\}, \varnothing) = (\varnothing, \varnothing, \varnothing),$$

$$A \cup B = (\{\text{High}, \text{Low}, \text{Med}\}, \{\text{Good}, \text{Poor}, \text{Fair}\}, \{\text{FT}, \text{Unemp}, \text{PT}\}) = (J_1, J_2, J_3).$$

Thus

$$\underline{F(A \cap B)} = \underline{\varnothing} = \varnothing, \quad \overline{F(A \cap B)} = \overline{\varnothing} = \varnothing,$$

$$\underline{F(A \cup B)} = \underline{X} = X, \quad \overline{F(A \cup B)} = \overline{X} = X.$$

This example illustrates how an n-SuperHyperrough Set captures multi-attribute uncertainties and region-based equivalences in a real-world loan screening scenario.

3. Results: New Graph Classes

As the main results of this paper, we define the classes of Hyperrough Graphs and *n*–SuperHyperrough Graphs.

3.1. Hyperrough Graphs

We now present the formal definition of Hyperrough Graphs.

Definition 3.1 (Hyperrough Graph)**.** *Let $\mathscr{I} = (U, A)$ be an information system with attributes $A = \{T_1, \ldots, T_n\}$ and induced equivalence relation $R \subseteq U \times U$. Set $J = J_1 \times \cdots \times J_n$, where each J_i is the domain of attribute T_i. A Hyperrough Graph is a quadruple*

$$H = (U, E, F, R)$$

where:

1. *$F : J \to \mathscr{P}(U)$ assigns each attribute-tuple $a \in J$ a subset $F(a) \subseteq U$.*
2. *For each $a \in J$, define*

$$\underline{F(a)} = \{x \in U \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in U \mid [x]_R \cap F(a) \neq \varnothing\}.$$

3. *The hyperrough-membership function is*

$$\mu_H : U \to [0,1]^J, \qquad \mu_H(x) = (\mu^A_{F(a)}(x))_{a \in J},$$

where $\mu^A_{F(a)}(x) = |[x]_R \cap F(a)| / |[x]_R|$.

4. *The edge set $E \subseteq \binom{U}{2}$ is given by*

$$\{u, v\} \in E \iff \max_{a \in J} \{\mu^A_{F(a)}(u), \mu^A_{F(a)}(v)\} > 0.$$

Example 3.2 (Hyperrough Graph for Consumer Preferences)**.** *Consumer preferences refer to individual choices, tastes, and priorities that influence purchasing decisions based on satisfaction, utility, and needs (cf.[51–53]). Consider a small market of four consumers*

$$U = \{u_1, u_2, u_3, u_4\}.$$

We record two binary preferences:

$$A = \{LikesA, LikesB\}, \quad Dom(LikesA) = Dom(LikesB) = \{Y,N\}.$$

Thus

$$J = \{Y,N\} \times \{Y,N\} = \{(Y,Y),(Y,N),(N,Y),(N,N)\}.$$

Define an equivalence relation R by consumer loyalty tier:

$$[u_1]_R = \{u_1,u_2\}, \quad [u_3]_R = \{u_3,u_4\}.$$

We assign each preference-tuple $a \in J$ the subset of consumers who expressed that combination:

$$F(Y,Y) = \{u_1,u_3\}, \quad F(Y,N) = \{u_1,u_2\},$$
$$F(N,Y) = \{u_3,u_4\}, \quad F(N,N) = \{u_2,u_4\}.$$

For each $a \in J$, the lower and upper approximations are:

$$\underline{F(Y,N)} = \{u_1,u_2\}, \quad \overline{F(Y,N)} = \{u_1,u_2\},$$

$$\underline{F(Y,Y)} = \varnothing, \quad \overline{F(Y,Y)} = U,$$

and similarly for (N,Y) and (N,N).
The hyperrough-membership of each consumer u is the vector

$$\mu_H(u) = \left(\mu^A_{F(a)}(u)\right)_{a\in J},$$

where for example

$$\mu^A_{F(Y,Y)}(u_1) = \frac{\left|[u_1]_R \cap \{u_1,u_3\}\right|}{\left|[u_1]_R\right|} = \frac{1}{2}, \quad \mu^A_{F(Y,N)}(u_1) = \frac{|\{u_1,u_2\} \cap \{u_1,u_2\}|}{2} = 1.$$

Finally, the Hyperrough Graph $H = (U,E,F,R)$ has edge set

$$E = \left\{\{u,v\} \subseteq U \mid \max_{a\in J}\{\mu^A_{F(a)}(u),\, \mu^A_{F(a)}(v)\} > 0\right\}.$$

In this example every consumer has at least one nonzero membership, so E is the complete graph on U.

Theorem 3.3. *A Hyperrough Graph $H = (U,E,F,R)$ carries the structure of a Hyperrough Set:*

$$\left\{(\underline{F(a)}, \overline{F(a)}) \mid a \in J\right\}.$$

Proof. By construction, for each $a \in J$ the pair $(\underline{F(a)}, \overline{F(a)})$ satisfies all axioms of a Hyperrough Set:

- $\underline{F(a)} \subseteq \overline{F(a)}$.
- If $\overline{F(a)} = \varnothing$ or $F(a) = U$, then both approximations coincide with $\varnothing$ or U, respectively.
- Monotonicity under union and intersection of tuples follows from properties of lower/upper approximations in rough set theory.

The vector-valued map μ_H then encodes exactly the rough-membership components required by the Hyperrough-Set formalism. □

Theorem 3.4. *Every ordinary rough graph (V,E,μ) is obtained as a Hyperrough Graph with a single attribute.*

Proof. Let (V,E,μ) be a rough graph. Take one attribute T with domain J_1 and set $J = J_1$. Choose $a_0 \in J$ and define

$$F(a) = \begin{cases} V, & a = a_0, \\ \varnothing, & a \neq a_0. \end{cases}$$

Then $\mu^A_{F(a_0)}(x) = 1$ and $\mu^A_{F(a)}(x) = 0$ for $a \neq a_0$. Hence the hyperrough-membership collapses to the scalar $\mu(x)$, and the edge rule $\max_a\{\dots\} > 0$ reduces exactly to the rough-graph criterion "$\max\{\mu(u),\mu(v)\} > 0$." This recovers (V,E,μ) as a Hyperrough Graph. □

Theorem 3.5 (Monotonicity of Approximations)**.** *Let $H = (U,E,F,R)$ be a Hyperrough Graph with attribute-tuple domain J. If $a,b \in J$ satisfy*

$$F(a) \subseteq F(b),$$

then

$$\underline{F(a)} \subseteq \underline{F(b)}, \quad \overline{F(a)} \subseteq \overline{F(b)}.$$

Proof. Suppose $x \in \underline{F(a)}$. By definition, $[x]_R \subseteq F(a)$. Since $F(a) \subseteq F(b)$, we conclude $[x]_R \subseteq F(b)$, hence $x \in \underline{F(b)}$. This shows $\underline{F(a)} \subseteq \underline{F(b)}$.
Next, suppose $x \in \overline{F(a)}$. Then $[x]_R \cap F(a) \neq \varnothing$. Because $F(a) \subseteq F(b)$, it follows $[x]_R \cap F(b) \neq \varnothing$, so $x \in \overline{F(b)}$. Thus $\overline{F(a)} \subseteq \overline{F(b)}$. □

Theorem 3.6 (Intersection and Union Properties)**.** *For any $a, b \in J$, let*

$$a \wedge b = (a_1 \wedge b_1, \ldots, a_n \wedge b_n), \quad a \vee b = (a_1 \vee b_1, \ldots, a_n \vee b_n)$$

denote the component-wise meet and join in the Cartesian product J. Then:

$$\underline{F(a \wedge b)} = \underline{F(a)} \cap \underline{F(b)}, \quad \overline{F(a \vee b)} = \overline{F(a)} \cup \overline{F(b)}.$$

Proof. Let $x \in \underline{F(a \wedge b)}$. By definition, $[x]_R \subseteq F(a \wedge b) = F(a) \cap F(b)$. Hence $[x]_R \subseteq F(a)$ and $[x]_R \subseteq F(b)$, so $x \in \underline{F(a)} \cap \underline{F(b)}$. Conversely, if $x \in \underline{F(a)} \cap \underline{F(b)}$, then $[x]_R \subseteq F(a)$ and $[x]_R \subseteq F(b)$, so $[x]_R \subseteq F(a) \cap F(b)$, giving $x \in \underline{F(a \wedge b)}$.

Next, let $x \in \overline{F(a \vee b)}$. Then $[x]_R \cap (F(a) \cup F(b)) \neq \varnothing$, so either $[x]_R \cap F(a) \neq \varnothing$ or $[x]_R \cap F(b) \neq \varnothing$. In the first case $x \in \overline{F(a)}$, and in the second $x \in \overline{F(b)}$. Thus $x \in \overline{F(a)} \cup \overline{F(b)}$. The reverse inclusion is immediate from $F(a) \subseteq F(a \vee b)$ and $F(b) \subseteq F(a \vee b)$. □

Theorem 3.7 (Crisp α-Threshold Subgraphs)**.** *Fix any $\alpha \in (0, 1]$. Define the α-level subgraph*

$$G^{\alpha} = (U, E^{\alpha}), \qquad E^{\alpha} = \left\{ \{u, v\} \in \binom{U}{2} \;\middle|\; \max_{a \in J} \{\mu^{A}_{F(a)}(u), \mu^{A}_{F(a)}(v)\} \geq \alpha \right\}.$$

Then G^{α} is an ordinary (crisp) undirected graph. Moreover, if $0 < \alpha_1 \leq \alpha_2 \leq 1$, then

$$E^{\alpha_2} \subseteq E^{\alpha_1},$$

so the family $\{G^{\alpha}\}_{\alpha \in (0,1]}$ is nested.

Proof. By construction, $E^{\alpha} \subseteq \binom{U}{2}$, and the edge-condition is symmetric in u, v, so G^{α} is a well-defined undirected graph.
If $\alpha_1 \leq \alpha_2$, then

$$\{u, v\} \in E^{\alpha_2} \implies \max\{\mu(u), \mu(v)\} \geq \alpha_2 \implies \max\{\mu(u), \mu(v)\} \geq \alpha_1 \implies \{u, v\} \in E^{\alpha_1},$$

where we abbreviate $\mu(x) = \max_{a \in J} \mu^{A}_{F(a)}(x)$. Hence $E^{\alpha_2} \subseteq E^{\alpha_1}$. □

3.2. n–SuperHyperrough Graphs

We now present the formal definition of n–SuperHyperrough Graphs.

Definition 3.8 (n-SuperHyperrough Graph)**.** *An* n-SuperHyperrough Graph *is a quadruple*

$$G^{(n)} = \big(U, E, F, R\big)$$

where:

1. *$F : J^{(n)} \to \mathscr{P}(U)$ assigns to each tuple $a = (A_1, \ldots, A_k) \in J^{(n)}$ a subset $F(a) \subseteq U$.*
2. *For each $a \in J^{(n)}$, define its* lower *and* upper *approximations by*

$$\underline{F(a)} = \{x \in U \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in U \mid [x]_R \cap F(a) \neq \varnothing\}.$$

3. *The* superhyperrough-membership *function is*

$$\mu^{(n)} : U \longrightarrow [0, 1]^{J^{(n)}}, \qquad \mu^{(n)}(x) = \big(\mu^{A}_{F(a)}(x)\big)_{a \in J^{(n)}},$$

   where $\mu^{A}_{F(a)}(x) = \dfrac{|[x]_R \cap F(a)|}{|[x]_R|}$ is the classical rough-membership of x in $F(a)$.
4. *The edge set $E \subseteq \binom{U}{2}$ is*

$$\{u, v\} \in E \quad \Longleftrightarrow \quad \max_{a \in J^{(n)}} \big\{\mu^{A}_{F(a)}(u), \mu^{A}_{F(a)}(v)\big\} > 0.$$

Example 3.9 (2–SuperHyperrough Graph in Environmental Monitoring)**.** *Environmental monitoring involves systematic collection and analysis of data on air, water, soil, and ecosystems to track changes and ensure safety (cf.[54–56]). Consider a network of six environmental sensors:*

$$U = \{s_1, s_2, s_3, s_4, s_5, s_6\}.$$

Each sensor has two attributes:

$$T_1 : \mathit{Zone} \in \{\text{North}, \text{South}\}, \quad T_2 : \mathit{Type} \in \{\text{Temp}, \text{Humid}\}.$$

Thus

$$J_1 = \{\text{North}, \text{South}\}, \; J_2 = \{\text{Temp}, \text{Humid}\},$$

and the 2-fold superhyper domain is

$$J^{(2)} = \mathscr{P}(J_1) \times \mathscr{P}(J_2) = \big\{\emptyset, \{\text{North}\}, \{\text{South}\}, \{\text{North}, \text{South}\}\big\} \times \big\{\emptyset, \{\text{Temp}\}, \{\text{Humid}\}, \{\text{Temp}, \text{Humid}\}\big\}.$$

We group sensors by hardware model:

$$[s_1]_R = \{s_1, s_2\}, \quad [s_3]_R = \{s_3, s_4\}, \quad [s_5]_R = \{s_5, s_6\}.$$

Define $F : J^{(2)} \to \mathscr{P}(U)$ *on a few representative tuples:*

$$F\big(\{\text{North}\}, \{\text{Temp}\}\big) = \{s_1, s_3\}, \qquad F\big(\{\text{South}\}, \{\text{Temp}\}\big) = \{s_2, s_4\},$$
$$F\big(\{\text{North}, \text{South}\}, \{\text{Humid}\}\big) = \{s_5, s_6\}, \quad F\big(\{\text{North}\}, \{\text{Temp}, \text{Humid}\}\big) = \{s_1\}.$$

All other combinations are mapped to $\varnothing$.
For each $a \in J^{(2)}$*, the lower and upper approximations in* $H = (U, E, F, R)$ *are*

$$\underline{F(a)} = \{x \in U \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in U \mid [x]_R \cap F(a) \neq \varnothing\}.$$

For example, for $a = (\{\text{North}\}, \{\text{Temp}\})$*:*

$$\underline{F(a)} = \{s_1\}, \quad \overline{F(a)} = \{s_1, s_2, s_3\},$$

since $[s_1]_R = \{s_1, s_2\} \subseteq \{s_1, s_3\}$ *is false for* s_2 *but true for* s_1*, and* $[s_3]_R \cap \{s_1, s_3\} = \{s_3\} \neq \varnothing$.
Each sensor's superhyperrough membership is the vector

$$\mu^{(2)}(x) = \big(\mu^A_{F(a)}(x)\big)_{a \in J^{(2)}}, \quad \mu^A_{F(a)}(x) = \frac{|[x]_R \cap F(a)|}{|[x]_R|}.$$

For s_1 *and* $a = (\{\text{North}\}, \{\text{Temp}\})$*:*

$$\mu^A_{F(a)}(s_1) = \frac{\big|\{s_1, s_2\} \cap \{s_1, s_3\}\big|}{2} = \tfrac{1}{2}.$$

Finally, the edge set is

$$E = \Big\{\{u, v\} \subseteq U \;\Big|\; \max_{a \in J^{(2)}} \{\mu^A_{F(a)}(u), \mu^A_{F(a)}(v)\} > 0\Big\}.$$

In this example every sensor has at least one nonzero membership component, so E *is the complete graph on* U.

Example 3.10 (3–SuperHyperrough Graph for Employee Collaboration)**.** *Employee collaboration is the process where team members work together, share ideas, and coordinate tasks to achieve common organizational goals (cf.[57–59]). Let* $U = \{e_1, e_2, e_3, e_4, e_5\}$ *be five employees. We record three categorical attributes:*

$$T_1 : \textit{Department} \in \{\text{HR}, \text{Eng}\}, \quad T_2 : \textit{Seniority} \in \{\text{Jr}, \text{Sr}\}, \quad T_3 : \textit{Location} \in \{\text{NY}, \text{SF}\}.$$

Thus

$$J_1 = \{\text{HR}, \text{Eng}\}, \quad J_2 = \{\text{Jr}, \text{Sr}\}, \quad J_3 = \{\text{NY}, \text{SF}\},$$

and the 3-fold superhyper domain is

$$J^{(3)} = \mathscr{P}(J_1) \times \mathscr{P}(J_2) \times \mathscr{P}(J_3).$$

Define the indiscernibility relation R *by project teams:*

$$[e_1]_R = \{e_1, e_3\}, \quad [e_2]_R = \{e_2, e_4\}, \quad [e_5]_R = \{e_5\}.$$

We specify $F : J^{(3)} \to \mathscr{P}(U)$ *on the following tuples (all others map to* $\varnothing$*):*

$$F\big(\{\text{HR}\}, \{\text{Jr}\}, \{\text{NY}\}\big) = \{e_1, e_2\}, \qquad F\big(\{\text{Eng}\}, \{\text{Sr}\}, \{\text{SF}\}\big) = \{e_3, e_4\},$$
$$F\big(\{\text{HR}, \text{Eng}\}, \{\text{Jr}\}, \{\text{NY}, \text{SF}\}\big) = \{e_5\}.$$

For each $a \in J^{(3)}$*, the lower and upper approximations are*

$$\underline{F(a)} = \{x \in U \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in U \mid [x]_R \cap F(a) \neq \varnothing\}.$$

For example, for $a = (\{\text{HR}\}, \{\text{Jr}\}, \{\text{NY}\})$*:*

$$\underline{F(a)} = \{e_1\}, \qquad \overline{F(a)} = \{e_1, e_2, e_3\},$$

since $[e_1]_R = \{e_1, e_3\} \subseteq \{e_1, e_2\}$ *is false (so* $e_3 \notin \underline{F(a)}$*) but* $[e_1]_R \cap \{e_1, e_2\} \neq \varnothing$ *and* $[e_3]_R \cap \{e_1, e_2\} = \{e_1\}$.
The superhyperrough-membership vector of each $e \in U$ *is*

$$\mu^{(3)}(e) = \big(\mu^A_{F(a)}(e)\big)_{a \in J^{(3)}}, \quad \mu^A_{F(a)}(e) = \frac{|[e]_R \cap F(a)|}{|[e]_R|}.$$

For e_1 and the above a,

$$\mu^A_{F(a)}(e_1) = \frac{|\{e_1,e_3\} \cap \{e_1,e_2\}|}{2} = \tfrac{1}{2}.$$

Finally, the edge set of the 3–SuperHyperrough Graph $G^{(3)} = (U,E,F,R)$ is

$$E = \big\{\{u,v\} \subseteq U \mid \max_{a \in J^{(3)}} \{\mu^A_{F(a)}(u), \mu^A_{F(a)}(v)\} > 0\big\}.$$

In this example, every employee has at least one nonzero membership value, so E is the complete graph on U.

Theorem 3.11. *The family $\big\{(\underline{F(a)}, \overline{F(a)}) \mid a \in J^{(n)}\big\}$ equips U with the structure of an n-SuperHyperrough Set.*

Proof. By construction, each pair $(\underline{F(a)}, \overline{F(a)})$ satisfies:

- $\underline{F(a)} \subseteq \overline{F(a)}$.
- If $F(a) = \varnothing$ or $F(a) = U$, then $\underline{F(a)} = \overline{F(a)} = \varnothing$ or U, respectively.
- Monotonicity under set-theoretic operations on the components of $a \in J^{(n)}$ follows by induction on n from the properties of lower and upper approximations in classical rough set theory.

Hence the mapping $a \mapsto (\underline{F(a)}, \overline{F(a)})$ satisfies the axioms of an n-SuperHyperrough Set. □

Theorem 3.12. *Every Rough Graph and every Hyperrough Graph arise as special cases of an n-SuperHyperrough Graph:*

- *By choosing $k = 1$, J_1 a singleton, and $n \geq 1$, one recovers the original Rough Graph structure.*
- *By restricting $J^{(n)}$ to tuples composed of singleton subsets of each $J_i \subseteq \mathscr{P}^n(J_i)$, one recovers the Hyperrough Graph construction.*

Proof. **(Rough Graph)** Let (V,E,μ) be a rough graph. Take $k = 1$, $J_1 = \{*\}$, and set $n \geq 1$. Define $F(\{*\},\ldots,\{*\}) = V$. Then for all $x \in V$, $\mu^A_{F(\{*\},\ldots)}(x) = 1$, so $\mu^{(n)}(x) \equiv 1$, and the edge rule $\max\{\mu^{(n)}(u), \mu^{(n)}(v)\} > 0$ becomes $\max\{\mu(u), \mu(v)\} > 0$.

(Hyperrough Graph) Let (U,E,F_{hyp},R) be a hyperrough graph with $F_{\text{hyp}} \colon J_1 \times \cdots \times J_k \to \mathscr{P}(U)$. For $n \geq 1$, note that each $(j_1,\ldots,j_k) \in J_1 \times \cdots \times J_k$ corresponds to $\big(\{j_1\},\ldots,\{j_k\}\big) \in J^{(n)}$. Restrict the mapping F of the n-SuperHyperrough Graph to these singleton tuples by setting $\widetilde{F}(\{j_1\},\ldots,\{j_k\}) = F_{\text{hyp}}(j_1,\ldots,j_k)$ and ignore all other tuples. Then the resulting membership and edge-rules exactly match those of the original hyperrough graph. □

Theorem 3.13 (Monotonicity)**.** *Let $a,b \in J^{(n)}$ satisfy $F(a) \subseteq F(b)$. Then*

$$\underline{F(a)} \subseteq \underline{F(b)}, \quad \overline{F(a)} \subseteq \overline{F(b)}.$$

Proof. If $x \in \underline{F(a)}$, then $[x]_R \subseteq F(a)$. Since $F(a) \subseteq F(b)$, we have $[x]_R \subseteq F(b)$, so $x \in \underline{F(b)}$. Similarly, $x \in \overline{F(a)}$ means $[x]_R \cap F(a) \neq \varnothing$. Because $F(a) \subseteq F(b)$, $[x]_R \cap F(b) \neq \varnothing$, hence $x \in \overline{F(b)}$. □

Theorem 3.14 (Intersection and Union of Tuples)**.** *For any $a,b \in J^{(n)}$, define component-wise meet $a \wedge b$ and join $a \vee b$. Then*

$$\underline{F(a \wedge b)} = \underline{F(a)} \cap \underline{F(b)}, \quad \overline{F(a \vee b)} = \overline{F(a)} \cup \overline{F(b)}.$$

Proof. If $x \in F(a \wedge b)$, then $[x]_R \subseteq F(a \wedge b) = F(a) \cap F(b)$, so $[x]_R \subseteq F(a)$ and $[x]_R \subseteq F(b)$, hence $x \in \underline{F(a)} \cap \underline{F(b)}$. Conversely, if $x \in \underline{F(a)} \cap \underline{F(b)}$, then $[x]_R \subseteq F(a) \cap F(b)$, giving $x \in \underline{F(a \wedge b)}$.

Similarly, $x \in \overline{F(a \vee b)}$ means $[x]_R \cap (F(a) \cup F(b)) \neq \varnothing$, so $[x]_R \cap F(a) \neq \varnothing$ or $[x]_R \cap F(b) \neq \varnothing$, i.e. $x \in \overline{F(a)} \cup \overline{F(b)}$. The reverse inclusion follows from $F(a) \subseteq F(a \vee b)$ and $F(b) \subseteq F(a \vee b)$. □

Theorem 3.15 (Crisp α-Level Subgraphs)**.** *For any threshold $\alpha \in (0,1]$, define the subgraph*

$$G^{(n)}_\alpha = \big(U, E_\alpha\big), \quad E_\alpha = \big\{\{u,v\} \mid \max_{a \in J^{(n)}} \{\mu^A_{F(a)}(u), \mu^A_{F(a)}(v)\} \geq \alpha\big\}.$$

Then $G^{(n)}_\alpha$ is an ordinary undirected graph, and if $\alpha_1 \leq \alpha_2$ then $E_{\alpha_2} \subseteq E_{\alpha_1}$.

Proof. By construction $E_\alpha \subseteq \binom{U}{2}$ and is symmetric, so $G^{(n)}_\alpha$ is an undirected graph. If $\alpha_1 \leq \alpha_2$ and $\{u,v\} \in E_{\alpha_2}$, then $\max\{\mu^A_{F(a)}(u), \mu^A_{F(a)}(v)\} \geq \alpha_2 \geq \alpha_1$, so $\{u,v\} \in E_{\alpha_1}$. Hence $E_{\alpha_2} \subseteq E_{\alpha_1}$. □

Theorem 3.16 (Closure under Tuplewise Union and Intersection)**.** *Let $G^{(n)}_1 = (U,E_1,F_1,R)$ and $G^{(n)}_2 = (U,E_2,F_2,R)$ be two n–SuperHyperrough Graphs on the same information system (U,A). Define new mappings*

$$(F_1 \wedge F_2)(a) = F_1(a) \cap F_2(a), \quad (F_1 \vee F_2)(a) = F_1(a) \cup F_2(a), \quad \forall a \in J^{(n)}.$$

Let $E_\wedge$ and $E_\vee$ be the corresponding edge-sets determined by the usual membership-threshold rule. Then

$$G^{(n)}_\wedge = (U, E_\wedge, F_1 \wedge F_2, R), \quad G^{(n)}_\vee = (U, E_\vee, F_1 \vee F_2, R)$$

are also n–SuperHyperrough Graphs. Moreover, for every $a \in J^{(n)}$,

$$\underline{(F_1 \wedge F_2)(a)} = \underline{F_1(a)} \cap \underline{F_2(a)}, \qquad \overline{(F_1 \wedge F_2)(a)} = \overline{F_1(a)} \cap \overline{F_2(a)},$$
$$\underline{(F_1 \vee F_2)(a)} = \underline{F_1(a)} \cup \underline{F_2(a)}, \qquad \overline{(F_1 \vee F_2)(a)} = \overline{F_1(a)} \cup \overline{F_2(a)}.$$

Proof. Fix any $a \in J^{(n)}$. Since rough approximations commute with finite set-union and set-intersection,

$$[x]_R \subseteq F_1(a) \cap F_2(a) \iff [x]_R \subseteq F_1(a) \text{ and } [x]_R \subseteq F_2(a),$$

hence $\underline{(F_1 \wedge F_2)(a)} = \underline{F_1(a)} \cap \underline{F_2(a)}$. A similar argument shows $\overline{(F_1 \wedge F_2)(a)} = \overline{F_1(a)} \cap \overline{F_2(a)}$.
For the union case,

$$[x]_R \cap \big(F_1(a) \cup F_2(a)\big) \neq \varnothing \iff [x]_R \cap F_1(a) \neq \varnothing \text{ or } [x]_R \cap F_2(a) \neq \varnothing,$$

yielding $\overline{(F_1 \vee F_2)(a)} = \overline{F_1(a)} \cup \overline{F_2(a)}$. The formula for the lower union follows from duality of lower/upper approximations under complementation.
Finally, the edge-set definitions based on $\max\{\mu^A_{F_i(a)}(u), \mu^A_{F_i(a)}(v)\} > 0$ inherit closure properties under $\wedge$ and $\vee$, so $G^{(n)}_\wedge$ and $G^{(n)}_\vee$ satisfy all requirements of an n–SuperHyperrough Graph. □

Theorem 3.17 (Attribute Marginalization)**.** *Let $G^{(n)} = (U, E, F, R)$ be an n–SuperHyperrough Graph on attributes $\{T_1, \ldots, T_k\}$. Fix an index i and denote*

$$A' = A \setminus \{T_i\}, \quad J'^{(n)} = \prod_{j \neq i} \mathscr{P}^n(J_j).$$

Define

$$F' \colon J'^{(n)} \longrightarrow \mathscr{P}(U), \qquad F'(b) = \bigcup_{A_i \subseteq J_i} F\big(b_1, \ldots, b_{i-1}, A_i, b_{i+1}, \ldots, b_k\big).$$

Let E' be the edge-set determined by the same membership-threshold rule applied to F'. Then

$$G'^{(n)} = \big(U, E', F', R\big)$$

is an n–SuperHyperrough Graph on the reduced attribute set A'. Moreover, for each $b \in J'^{(n)}$,

$$\underline{F'(b)} = \bigcup_{A_i \subseteq J_i} \underline{F(b, A_i)}, \quad \overline{F'(b)} = \bigcup_{A_i \subseteq J_i} \overline{F(b, A_i)}.$$

Proof. By definition,

$$\underline{F'(b)} = \{x \in U \mid [x]_R \subseteq \bigcup_{A_i} F(b, A_i)\}.$$

Since $[x]_R \subseteq \bigcup_{A_i} F(b,A_i)$ if and only if $[x]_R \subseteq F(b,A_i)$ for some A_i, we get $\underline{F'(b)} = \bigcup_{A_i} \underline{F(b,A_i)}$. Dually,

$$\overline{F'(b)} = \{x \in U \mid [x]_R \cap \bigcup_{A_i} F(b,A_i) \neq \varnothing\} = \bigcup_{A_i} \{x \mid [x]_R \cap F(b,A_i) \neq \varnothing\} = \bigcup_{A_i} \overline{F(b,A_i)}.$$

The edge condition $\max_{b \in J'^{(n)}} \mu^A_{F'(b)}(x) > 0$ follows from the fact that $\mu^A_{F'(b)}(x) = \sup_{A_i} \mu^A_{F(b,A_i)}(x)$, so E' is well-defined and $G'^{(n)}$ satisfies the n–SuperHyperrough Graph axioms. □

4. Results: Hyperrough Tree-Width

We introduce a new graph width parameter in this section.

4.1. Hyperrough Tree-Width

The definition of Hyperrough Tree-Width is presented below.

Definition 4.1 (Hyperrough Tree Decomposition)**.** *A* hyperrough tree decomposition *of H is a pair*

$$\big(T, \{B_t\}_{t \in V(T)}\big),$$

where T is a tree and each $B_t \subseteq U$ (a bag*) satisfies the usual tree-decomposition axioms:*

(HTD1) $\bigcup_{t \in V(T)} B_t = U$.
(HTD2) For every edge $\{u, v\} \in E$, there is $t \in V(T)$ with $\{u, v\} \subseteq B_t$.
(HTD3) For each $u \in U$, the set $\{t \in V(T) \mid u \in B_t\}$ induces a connected subtree of T.

For each attribute-tuple $a \in J$, define:

$$\ell_t(a) = \big|\{u \in B_t \mid [x]_R \subseteq F(a)\}\big|, \quad u_t(a) = \big|\{u \in B_t \mid [x]_R \cap F(a) \neq \varnothing\}\big|.$$

Then the lower width *and* upper width *of the decomposition at a are*

$$\ell(a) = \min_{(T,B)} \max_{t \in V(T)} \big(\ell_t(a) - 1\big), \quad u(a) = \min_{(T,B)} \max_{t \in V(T)} \big(u_t(a) - 1\big).$$

The hyperrough tree-width *of H is the mapping*

$$a \longmapsto \big(\ell(a), u(a)\big) \in \{(p,q) \in \mathbb{N}^2 \mid p \leq q\}.$$

Example 4.2 (Hyperrough Tree Decomposition of a Consumer-Preference Hyperrough Graph)**.** *Let $H = (U, E, F, R)$ be the Hyperrough Graph described in Example, with*

$$U = \{u_1, u_2, u_3, u_4\}, \quad R \text{ partitions } U \text{ into } \{u_1, u_2\} \text{ and } \{u_3, u_4\},$$

and attribute set $A = \{\text{LikesA}, \text{LikesB}\}$ giving

$$J = \{Y, N\} \times \{Y, N\} = \{(Y,Y), (Y,N), (N,Y), (N,N)\},$$

and

$$F(Y,N) = \{u_1, u_2\}, \quad F(Y,Y) = \{u_1, u_3\}, \quad F(N,Y) = \{u_3, u_4\}, \quad F(N,N) = \{u_2, u_4\}.$$

We choose the following tree decomposition $(T, \{B_t\})$ of the underlying crisp graph:

$$T: \quad t_1 - t_2, \qquad B_{t_1} = \{u_1, u_2, u_3\}, \quad B_{t_2} = \{u_2, u_3, u_4\}.$$

One readily verifies (HTD1)–(HTD3).

***Computation for** $a = (Y,N)$:*

$$\underline{F(a)} = \{x \mid [x]_R \subseteq \{u_1, u_2\}\} = \{u_1, u_2\}, \quad \overline{F(a)} = \{u_1, u_2\}.$$

For each bag:

$$\ell_{t_1}(a) = \big|\{x \in B_{t_1} \mid [x]_R \subseteq F(a)\}\big| = 2, \quad \ell_{t_2}(a) = 1,$$

$$u_{t_1}(a) = \big|\{x \in B_{t_1} \mid [x]_R \cap F(a) \neq \varnothing\}\big| = 2, \quad u_{t_2}(a) = 1.$$

Hence

$$\ell(a) = \max\{2-1, 1-1\} = 1, \quad u(a) = \max\{2-1, 1-1\} = 1.$$

***Computation for** $a' = (Y,Y)$:*

$$\underline{F(a')} = \{x \mid [x]_R \subseteq \{u_1, u_3\}\} = \varnothing, \quad \overline{F(a')} = U.$$

Thus

$$\ell_{t_1}(a') = 0,\ \ell_{t_2}(a') = 0 \implies \ell(a') = \max\{0-1, 0-1\} = -1 \mapsto 0,$$

$$u_{t_1}(a') = 3,\ u_{t_2}(a') = 3 \implies u(a') = \max\{3-1, 3-1\} = 2.$$

Therefore the hyperrough tree-width *mapping for H under this decomposition is*

$$(Y,N) \mapsto (1,1), \quad (Y,Y) \mapsto (0,2), \quad \textit{etc.}$$

which shows how each attribute-tuple $a \in J$ yields a pair $\big(\ell(a), u(a)\big)$ in $\mathbb{N}^2$.

Theorem 4.3. *The collection*

$$\big\{\big(\ell(a), u(a)\big) \mid a \in J\big\}$$

satisfies the axioms of a Hyperrough Set on J: $\ell(a) \leq u(a)$ for all a, $(\ell(a), u(a)) = (0,0)$ if $F(a) = \varnothing$ or $F(a) = U$, and monotonicity under joins and meets of tuples in J.

Proof. By construction, for each $a \in J$ we have $\ell(a) \leq u(a)$. If $F(a) = \varnothing$ then no vertex can satisfy $[x]_R \subseteq F(a)$ or intersect $F(a)$, so $\ell(a) = u(a) = 0$. If $F(a) = U$, every equivalence class is contained in $F(a)$, and hence $\ell_t(a) = |B_t|$ and $u_t(a) = |B_t|$ for all t, so again $\ell(a) = u(a) = \min_{(T,B)}(\max_t(|B_t| - 1))$, which is the classical tree-width of the underlying crisp graph. Finally, ordering on J by component-wise inclusion (i.e. joins and meets) carries over to inclusion relations among the sets $F(a)$, and hence to the lower/upper approximations. Standard arguments from rough-set theory show that ℓ and u respect these monotonicities, completing the verification of the Hyperrough-Set axioms. □

Theorem 4.4. *When $|A| = 1$, the hyperrough tree-width reduces exactly to the* rough tree-width *of the corresponding Rough Graph.*

Proof. With a single attribute T_1, we have $J = J_1$. Writing $a \in J_1$, the definitions of $\ell(a)$ and $u(a)$ coincide with those of the lower and upper widths in the rough tree decomposition (Definition 2.5). Hence the hyperrough tree-width mapping $a \mapsto (\ell(a), u(a))$ collapses to the rough tree-width scalar. □

Theorem 4.5 (Monotonicity of Widths)**.** *If $a, b \in J$ satisfy $F(a) \subseteq F(b)$, then*

$$\ell(a) \leq \ell(b), \quad u(a) \leq u(b).$$

Proof. Since $F(a) \subseteq F(b)$, for every bag B_t we have

$$\{x \in B_t \mid [x]_R \subseteq F(a)\} \ \subseteq\ \{x \in B_t \mid [x]_R \subseteq F(b)\},$$

so $\ell_t(a) \leq \ell_t(b)$ for all t. Taking the minimum over all decompositions yields $\ell(a) \leq \ell(b)$. A similar argument shows $\{x \in B_t \mid [x]_R \cap F(a) \neq \varnothing\} \subseteq \{x \in B_t \mid [x]_R \cap F(b) \neq \varnothing\}$, hence $u_t(a) \leq u_t(b)$ and thus $u(a) \leq u(b)$. □

Theorem 4.6 (Subgraph Correspondence)**.** *For each $a \in J$, let*

$$L(a) = \underline{F(a)}, \qquad U(a) = \overline{F(a)},$$

and denote by $H[L(a)]$ and $H[U(a)]$ the induced subgraphs of H on $L(a)$ and $U(a)$, respectively. Then

$$\ell(a) = \mathrm{tw}\big(H[L(a)]\big), \quad u(a) = \mathrm{tw}\big(H[U(a)]\big).$$

Proof. A tree decomposition $(T, \{B_t\})$ of H induces valid tree decompositions of the subgraphs $H[L(a)]$ and $H[U(a)]$ by restricting each bag to $B_t \cap L(a)$ and $B_t \cap U(a)$. In the former, a vertex $x \in L(a)$ appears in exactly those bags B_t for which $[x]_R \subseteq F(a)$, so the maximum bag-size minus one equals $\max_t(\ell_t(a) - 1)$. Taking the minimum over all decompositions recovers $\ell(a)$. An identical argument on $U(a)$ yields $u(a)$. □

Theorem 4.7 (Intersection and Union Behavior)**.** *For any $a, b \in J$, let $a \wedge b$ and $a \vee b$ denote their component-wise meet and join in J. Then*

$$\ell(a \wedge b) \leq \min\{\ell(a), \ell(b)\}, \quad u(a \vee b) \geq \max\{u(a), u(b)\}.$$

Proof. Since $F(a \wedge b) = F(a) \cap F(b)$, we have $\underline{F(a \wedge b)} \subseteq \underline{F(a)}$ and $\underline{F(a \wedge b)} \subseteq \underline{F(b)}$, so Theorem 4.5 implies $\ell(a \wedge b) \leq \ell(a)$ and $\ell(a \wedge b) \leq \ell(b)$. Similarly, $F(a \vee b) = F(a) \cup F(b)$ gives $\overline{F(a)} \subseteq \overline{F(a \vee b)}$ and $\overline{F(b)} \subseteq \overline{F(a \vee b)}$, so $u(a) \leq u(a \vee b)$ and $u(b) \leq u(a \vee b)$, hence $u(a \vee b) \geq \max\{u(a), u(b)\}$. □

5. Conclusion

In this paper, we extended the theoretical framework by proposing several new generalizations: the *Hyperrough Tree-width*, the *Hyperrough Graph*, and the *Superhyperrough Graph*. Future work may include computational experiments and further exploration of new graph classes based on these generalized structures.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Funding

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Author Contributions

The paper has been solely authored by the corresponding author at this stage.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Considerations

This work does not involve any experiments or studies involving human participants or animals, and therefore no ethical approvals were required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

References

[1] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[2] Zdzislaw Pawlak, Lech Polkowski, and Andrzej Skowron. Rough set theory. *KI*, 15(3):38–39, 2001.

[3] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[4] Zdzislaw Pawlak and Roman Słowinski. Decision analysis using rough sets. *International Transactions in Operational Research*, 1(1): 107–114, 1994.

[5] S Pavithra and A Manimaran. Uncertainty measure for z-soft covering based rough graphs with application. *Journal of Intelligent & Fuzzy Systems*, 44(4):5789–5802, 2023.

[6] R Nithya and K Anitha. Even vertex zeta-graceful labeling on rough graph. *arXiv preprint arXiv:2208.12047*, 2022.

[7] Andrzej Skowron and Cecylia Rauszer. The discernibility matrices and functions in information systems. In *Intelligent decision support: handbook of applications and advances of the rough sets theory*, pages 331–362. Springer, 1992.

[8] Saibal Majumder and Samarjit Kar. Multi-criteria shortest path for rough graph. *Journal of Ambient Intelligence and Humanized Computing*, 9:1835–1859, 2018.

[9] Takaaki Fujita. Neighborhood hyperrough set and neighborhood superhyperrough set. *Pure Mathematics for Theoretical Computer Science*, 5(1):34–47, 2025.

[10] Takaaki Fujita. Short introduction to rough, hyperrough, superhyperrough, treerough, and multirough set. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 394, 2025.

[11] Takaaki Fujita and Florentin Smarandache. A concise introduction to hyperfuzzy, hyperneutrosophic, hyperplithogenic, hypersoft, and hyperrough sets with practical examples. *Neutrosophic Sets and Systems*, 80:609–631, 2025.

[12] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. 2024.

[13] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[14] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[15] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[16] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[17] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[18] Neil Robertson and Paul D. Seymour. Graph minors. i. excluding a forest. *Journal of Combinatorial Theory, Series B*, 35(1):39–61, 1983.

[19] Takaaki Fujita. Improved version of short note: Exploring ideals in graph theory. *International Journal of Advanced Multidisciplinary Research and Studies*, 5(2):2043–2050, 2025.

[20] Takaaki Fujita. Ultrafilters and their dual relationship to tree-width in graph theory. *Asian Research Journal of Mathematics*, 21(1): 98–114, 2025.

[21] Hans L Bodlaender. A tourist guide through treewidth. *Acta cybernetica*, 11(1-2):1–21, 1993.

[22] Haim Kaplan and Ron Shamir. Pathwidth, bandwidth, and completion problems to proper interval graphs with small cliques. *SIAM Journal on Computing*, 25(3):540–561, 1996.

[23] Dariusz Dereniowski. From pathwidth to connected pathwidth. *SIAM Journal on Discrete Mathematics*, 26(4):1709–1732, 2012.

[24] Sang-il Oum and Paul Seymour. Testing branch-width. *Journal of Combinatorial Theory, Series B*, 97(3):385–393, 2007.

[25] Georgios Kontogeorgiou, Alexandros Leivaditis, Kostas I Psaromiligkos, Giannos Stamoulis, and Dimitris Zoros. Branchwidth is (1, g)-self-dual. *Discrete Applied Mathematics*, 350:1–9, 2024.

[26] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions: A survey. In *Mathematical Foundations of Computer Science 2001: 26th International Symposium, MFCS 2001 Mariánské Lázne, Czech Republic, August 27–31, 2001 Proceedings 26*, pages 37–57. Springer, 2001.

[27] Isolde Adler, Georg Gottlob, and Martin Grohe. Hypertree width and related hypergraph invariants. *European Journal of Combinatorics*, 28(8):2167–2181, 2007.

[28] Takaaki Fujita. Superhyperbranch-width and superhypertree-width. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 367, 2025.

[29] Takaaki Fujita. Superhypertree-length and superhypertree-breadth in superhypergraphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 41, 2025.

[30] Archontia C Giannopoulou, Michał Pilipczuk, Jean-Florent Raymond, Dimitrios M Thilikos, and Marcin Wrochna. Cutwidth: Obstructions and algorithmic aspects. *Algorithmica*, 81:557–588, 2019.

[31] Feodor F Dragan, Ekkehard Köhler, and Arne Leitert. Line-distortion, bandwidth and path-length of a graph. *Algorithmica*, 77: 686–713, 2017.

[32] Takaaki Fujita. Short note of rough tree-width from an information system. *ResearchGate*, 2024.

[33] Zdzislaw Pawlak, Jerzy Grzymala-Busse, Roman Slowinski, and Wojciech Ziarko. Rough sets. *Communications of the ACM*, 38(11): 88–95, 1995.

[34] R Aruna Devi and K Anitha. Construction of rough graph to handle uncertain pattern from an information system. *arXiv preprint arXiv:2205.10127*, 2022.

[35] R Noor, I Irshad, and I Javaid. Soft rough graphs. *arXiv preprint arXiv:1707.05837*, 2017.

[36] Takaaki Fujita and Florentin Smarandache. General, general weak, anti, balanced, and semi-neutrosophic graph. *Neutrosophic Sets and Systems*, 85:398–435, 2025.

[37] Neil Robertson and Paul D. Seymour. Graph minors. iii. planar tree-width. *Journal of Combinatorial Theory, Series B*, 36(1):49–64, 1984.

[38] Neil Robertson and Paul D. Seymour. Graph minors. x. obstructions to tree-decomposition. *Journal of Combinatorial Theory, Series B*, 52(2):153–190, 1991.

[39] NICOLE WEIN. Cyclewidth: An analogy of treewidth.

[40] Takaaki Fujita. Hyperfuzzy hypersoft set and hyperneutrosophic hypersoft set. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 247, 2025.

[41] Takaaki Fujita. Superhypersoft hyperrough set and superhypersoft superhyperrough set. *Preprint*, 2025.

[42] Takaaki Fujita. A study on hyperfuzzy hyperrough sets, hyperneutrosophic hyperrough sets, and hypersoft hyperrough sets with applications in cybersecurity. *Artificial Intelligence in Cybersecurity*, 2:14–36, 2025.

[43] Takaaki Fujita and Florentin Smarandache. A concise introduction to hyperfuzzy, hyperneutrosophic, hyperplithogenic, hypersoft, and hyperrough sets with practical examples. *Neutrosophic Sets and Systems*, 80:609–631, 2025.

[44] Takaaki Fujita. Note of indertermrough set and indetermhyperrough set. 2025.

[45] Takaaki Fujita. Forest hyperplithogenic set and forest hyperrough set. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 2025.

[46] Takaaki Fujita. Multipolar rough set, multipolar hyperrough set, multipolar soft set, and multipolar hypersoft set, 2025. Preprint.

[47] Takaaki Fujita. Probabilistic hyperrough set and covering hyperrough set. *Preprint*, 2025.

[48] Bryan Stanhouse and Larry Sherman. A note on information in the loan evaluation process. *The Journal of Finance*, 34(5):1263–1269, 1979.

[49] Sadig Mammadli. Fuzzy logic based loan evaluation system. *Procedia Computer Science*, 102:495–499, 2016.

[50] Kay Bryant. Alees: an agricultural loan evaluation expert system. *Expert systems with applications*, 21(2):75–85, 2001.

[51] Dale W Jorgenson and Lawrence J Lau. The structure of consumer preferences. In *Annals of Economic and Social Measurement, Volume 4, number 1*, pages 49–101. NBER, 1975.

[52] Fanchao Liao, Eric Molin, and Bert van Wee. Consumer preferences for electric vehicles: a literature review. *Transport Reviews*, 37(3): 252–275, 2017.

[53] Ravi Dhar and Itamar Simonson. The effect of the focus of comparison on consumer preferences. *Journal of Marketing Research*, 29 (4):430–440, 1992.

[54] Eric Biber. The problem of environmental monitoring. *U. Colo. L. Rev.*, 83:1, 2011.

[55] Janick F Artiola, Ian L Pepper, and Mark L Brusseau. *Environmental monitoring and characterization*. Academic Press, 2004.

[56] Gary M Lovett, Douglas A Burns, Charles T Driscoll, Jennifer C Jenkins, Myron J Mitchell, Lindsey Rustad, James B Shanley, Gene E Likens, and Richard Haeuber. Who needs environmental monitoring? *Frontiers in Ecology and the Environment*, 5(5):253–260, 2007.

[57] Robert L Cross, Roger D Martin, and Leigh M Weiss. Mapping the value of employee collaboration. *McKinsey Quarterly*, 3:28, 2006.

[58] Horace L Melton and Michael D Hartline. Employee collaboration, learning orientation, and new service development performance. *Journal of Service Research*, 16(1):67–81, 2013.

[59] Christian Meske, Tobias Brockmann, Konstantin Wilms, and Stefan Stieglitz. Gamify employee collaboration-a critical review of gamification elements in social software. *arXiv preprint arXiv:1606.01351*, 2016.